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A VES PRODUCTION FUNCTION AND URBAN LAND USE: A THEORETICAL AND EMPIRICAL INVESTIGATION

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A VES PRODUCTION FUNCTION AND URBAN LAND USE: A THEORETICAL AND EMPIRICAL INVESTIGATION

1. Introduction

In this study the variable elasticity of substitution (VES) production function developed by Revankar (16) is used to derive an urban land-use model and to investigate the impact of changing elasticity of substitution of land for capital on urban structure.[1]

Empirical studies on urban structure have often assumed a fixed value for the elasticity of substitution (σ) between capital and land. Initially urban models assumed very simple types of production functions such as the fixed and linear coefficient models with σ equal to zero and infinity, respectively. Muth (14) using a Cobb-Douglas production function with a σ equal to unity derived the conditions for a negative exponential function between density and distance. Mills (11) also using Cobb-Douglas functions developed additional insights into urban structure. The value of σ can range between zero and infinity; thus any empirical study using production functions which assumes any specific constant value of σ can lead to specification bias. Recently Fallis (3), Kau and Lee (5), and Muth (15) have used a constant elasticity of substitution (CES) production function to investigate the structure of a city. The CES production function arbitrarily constrains σ to be a constant and does not allow it to vary with a change in the factor input ratio.

As demonstrated by Hicks (4) and Allen (1), σ can vary depending on the factor combinations and output. Relative factor prices vary over a wide range within an urban area resulting in capital-land ratios (K/L) exemplified by comparing high-rise apartments to single-family dwellings. Thus a CES production function with a constant σ within a specified urban area will lead to specification bias. Revankar's VES production function incorporates the impact of the elasticity of substitution (σ) and the K/L ratio thus eliminating specification bias associated with changes in σ and the factor ratio.

In the second section, the importance of allowing changes in the elasticity of substitution within an urban area is discussed. Also, the bias in the density gradient resulting from omitting changes in σ is investigated. In the third section, a density gradient is derived using a VES production function with technological change. The bias of the rent gradient and density gradient associated with Cobb-Douglas production function is explored. In the fourth section, empirical estimates are presented for the elasticity of substitution associated with both CES and VES production functions. It is shown that the VES method can be used to take the change of σ into account. Finally, results of this paper are summarized.

2. Impact of the Elasticity of Substitution on the Density-Distance Function

The VES production function differs from the CES in its isoquant mapping. The elasticity of substitution for

the CES is the same at all points on the isoquant mapping, thus independent of output and the K/L ratio. The VES does not require that the elasticity of substitution be the same. Thus σ can vary with output and with changes in the K/L ratio. The elasticity of substitution can increase steadily from zero to infinity. In terms of the isoquants this is demonstrated by their becoming more flattened with increased output. Also it is shown later in this study that the isoquant becomes flatter with a decrease in the K/L ratio for the same output.

The properties of the VES function are especially important for studies of urban structure. Studies by Kau and Lee (6), Mills (12), and Muth (14) have demonstrated the existence of a downward slopping land-rent gradient, therefore the K/L ratio will decrease with increasing distance from the urban center.[2] Thus firms producing urban structures will become more capital intensive near the urban center.

Secondly, firms producing urban structures may have significantly different production functions relative to output. For example, firms producing high-rise condominiums near the urban center may have significantly different isoquant mappings than firms producing single-family dwellings on the outskirts of an urban area.[3]

Thirdly, due to natural and technological limitations, the ability to substitute capital for land decreases with increased capital intensity. For example, the heights of buildings are limited by space requirements for elevators and the size of the foundation. Therefore decreasing land

rents from the urban center and the corresponding decrease in the K/L ratio results in the elasticity of substitution increasing with distance.

Technological limitations, changing K/L ratios and different production functions make the Cobb-Douglas and CES production functions an inefficient tool for analyzing urban structure. Hence, the statistical studies on urban structure depending on a Cobb-Douglas [see Mills (11) and Muth (14)] or the CES production function [see Fallis (3), Koenker (8), Muth (15), and Tooze (19)] have bias empirical estimates.

The effect of variation in rentals and the elasticity of substitution on the intensity of residential land use can be investigated using the theoretical framework developed by Muth (14). The essence of the Muth model necessary for this paper is contained in the derivation of locational equilibrium of producers. Producers of housing combine land and capital in production to maximize profits. It is assumed that firms are competitive and that the price of capital is the same everywhere. Under these conditions rent per unit of land falls with distance.

Thus,

$$\pi = PQ(L,K) - rL - nK \tag{2.1}$$

where π is profit, P and Q the price per unit and output, respectively with inputs Land (L) and Capital (K) at unit prices r and n respectively.

Differentiating (2.1) and assuming that housing prices and wages vary only with distance (u) from the CBD

and assuming similar production functions for each unique location, we have, upon rearranging terms,

$$(2.2) \qquad \frac{\partial r_u}{r} = \frac{1}{\rho_L}\left[\frac{\partial P/\partial u}{P}\right] - \frac{\rho_K}{\rho_L}\left(\frac{\partial n/\partial u}{n}\right)$$

where

$$\rho_L = rL/PQ \quad \text{and} \quad \rho_K = nK/PQ$$

Equation (2.2) implies that the rents on land will be bid up by firms in locations where product price is high owing to favorable location.

The effects of variation in land rents and the elasticity of substitution on land use can be derived assuming that factor payments exhaust receipts. Thus,

$$(2.3) \qquad PQ = rL + nK$$

Dividing both sides by L, differentiating, with respect to u and using the definition of the elasticity of substitution and equation (2.2), the following equation is derived,

$$(2.4) \qquad \frac{\partial}{\partial u}\left(\frac{PQ}{L}\right)^* = \left(1 + \frac{\rho_K}{\rho_L}\sigma(u)\right)\frac{\partial P/\partial u}{P} \quad < \quad 0$$

where

* = the log value.

The PQ/L is a measure of the intensity of residential land use in terms of the value of housing produced per unit of land. The negative sign of equation (2.4) indicates the slope of the specified density function. The slope varies inversely with the relative importance of land and directly with the elasticity of substitution and $\partial P/\partial u$.

The variations in the slope of the curve which relates the log of the value of housing per square mile to distance from the CBD is obtained by differentiating equation (2.4) with respect to u. It is assumed that σ, the elasticity of substitution, is a function of distance u. The results are

$$(2.5) \qquad \frac{\partial^2}{\partial u^2}\left(\frac{PQ}{L}\right)^* = \left(1 + \frac{\rho_K}{\rho_L}\,\sigma(u)\right)\frac{\partial}{\partial u}\left(\frac{\partial P/\partial u}{P}\right)$$

$$+\left(\frac{\partial P/\partial u}{P}\right)\left[\sigma(\sigma-1)\,\frac{\rho_K}{\rho_L}\left(\frac{\partial r/\partial u}{r}\right)\right] + \left(\frac{\partial P/\partial u}{P}\right)\left[\frac{\rho_K}{\rho_L}\cdot\frac{\partial \sigma}{\partial u}\right]$$

It has been shown by Muth (14) that the slope of the log price-distance function tends to decline numerically with distance from the CBD. Thus,

$$(2.6) \qquad \frac{\partial}{\partial u}\left(\frac{\partial P/\partial u}{P}\right) \geq 0.$$

The second term of equation (2.5) represents the bias associated with assuming a Cobb-Douglas function. Empirically (demonstrated later in the paper) σ ranges from .09 to .93. The theoretical model indicates that $(\partial P/\partial u)/P$ and $\partial r/\partial u \leq 0$. Also ρ_K/ρ_L increases with distance[4] when $\sigma < 1$. The second term of equation (2.5) is negative.

The last term of equation (2.5) represents the bias associated with assuming a constant elasticity of substitution and provides the impact of changing elasticity of substitution on density. Since empirically it will be shown that σ increases

with distance and that since $\sigma < 1$, ρ_K/ρ_L increases with distance. Thus with $(\partial P/\partial u)/P < 0$, the last term is negative.

Both the second and third term indicate a density function concave to the origin, whereas the first term would produce a convex curve. The negative curvature of the second two terms tends to offset the positive curvature imparted to equation (2.5) by the log price-distance function. Hence, the density gradient may be approximately constant even though the price gradient declines with distance from the CBD. If the impact of variations in the elasticity of substitution is greater than the convexity imposed by the price-distance function then the density gradient may have a negative curvature so that population densities would decline less rapidly than negative exponentially with distance from the CBD. In this case holding price gradient constant an increase in demand for housing would increase the output of housing and population more rapidly in the outer parts of the city.

Urban models using Cobb-Douglas or CES production functions or the generalized Muth model (14) as represented by equations (2.2) (2.4) and the first two terms of (2.5) will be biased in their prediction of structural changes. Transformation procedures and statistical techniques have been developed by Kau and Lee (6) (7) to reduce the extent of this bias.

The purpose of this section has been to demonstrate the bias associated with restricting the elasticity of substitution to unity as well as with assuming a constant elasticity of substitution throughout an urban area. In the following section, the VES production function is used to derive the density gradient. The bias of the density gradient derived from the Cobb-Douglas production function is investigated.

3. The Density Gradient: A VES Approach

A production function which incorporates the impact of changing elasticities of substitution, capital-land ratios and technologies is the VES function as presented by Revankar (16) given by:

$$(3.1) \qquad V_{t(u)} = Ae^{\lambda t} K_{t(u)}^{\alpha(1-\delta\rho)} \cdot \left[L_{t(u)} + (\rho-1)(1+bt)K_{t}(u)\right]^{\alpha\delta\rho}$$

where $V_{t(u)}$ is the output of housing services u miles from the urban center, $K_{t(u)}$ and $L_{t(u)}$ represent inputs of capital and land, respectively, all in time period t. A, δ, ρ, α, and b are parameters with

$$A > 0, \ \alpha > 0$$

$$0 < \delta < 1, \ 0 \leq \delta\rho \leq 1$$

$$(3.2) \qquad L_t/K_t > \frac{(1-\rho)(1+bt)}{1-\delta\rho}$$

Neutral technical change is represented by $e^{\lambda t}$, with λ referred to as the neutrality parameter. Non-neutral technical change

is reflected by the term (1+bt) which is linear in time (t) with b referred to as the non-neutrality parameter.

A neutral technological change does not alter the marginal productivity ratio whereas a non-neutral technical change is capital-using (land-using) if it increases the marginal product of capital (land) relative to land (capital), holding the capital-land ratio constant.

The elasticity of substitution σ for the VES is

$$\sigma_t = \left[1 + \frac{(\rho-1)\ (1+bt)}{1-\delta\rho}\right]\left(\frac{K_t}{L_t}\right) \tag{3.3}$$

In this formulation σ_t varies linearly with time at each capital-land ratio. The presence of b measures the effect of bias in technical change in terms of its effect on the elasticity of substitution. Also σ_t varies with the capital-land ratio in each time period. Thus, as astutely demonstrated by Revankar (17), the elasticity of substitution has a two-way linear dependence: (i) on t for a given capital-land ratio and (ii) on the capital-land ratio for a given t. The VES function reduces to the Cobb-Douglas function when ρ=1 for all t.

A demand function for housing services at u, Xd(u) is defined as

$$X_t d(u) = B w^{\Theta 1} p_t(u)^{\Theta 2} \tag{3.4}$$

where B is a scale parameter and depends upon the units in which housing services are measured;

w = income for workers;

$p_t(u)$ = price of housing services at distance u, for a given time period;

Θ_1 = income elasticity; and

Θ_2 = price elasticity.

From equation (3.1) the relationship between the value of the marginal product and its rental rate for both land and capital [$r_t(u)$ and n_t] can be written as

$$(3.5) \qquad \frac{\partial V_t}{\partial L_t} = \alpha\delta\rho\left[\frac{V_t}{L_t+(\rho-1)K_t}\right] = \frac{r_t(u)}{P_t(u)}$$

$$(3.6) \qquad \frac{\partial V_t}{\partial K_t} = \alpha(1-\delta\rho)\ \frac{V_t}{K_t} + \alpha\delta\rho\cdot(\rho-1)(1+bt)\ \frac{V_t}{L_t+(\rho-1)K_t}$$

$$= \frac{n_t}{P_t(u)}$$

From equations (3.5) and (3.6) the marginal rate of substitution of capital for land, MS_t is

$$(3.7) \qquad MS_t = \frac{(\rho-1)\cdot(1+bt)}{\delta\rho} + \frac{1-\delta\rho}{\delta\rho}\left(\frac{L_t}{K_t}\right)$$

Note that MS_t changes linearly with time with capital-intensity remaining constant. If the technical bias is capital-using then the time rate of increase in MS_t at a fixed capital-intensity is

$$(3.8) \qquad \frac{dMS_t}{dt} = \frac{(\rho-1)b}{\delta\rho} > 0$$

The time elasticity of MS_t is

$$(3.9) \qquad E_t = \frac{(\rho-1)b}{(\rho-1)(1+bt) + (1-\delta\rho)\ L_t/K_t}$$

The elasticity is higher at greater capital-land ratios, i.e., higher K_t/L_t ratios lead to a greater rate of capital-using inventions.

From equations (3.5) and (3.6) we have

$$(3.10) \qquad L_t = \alpha P_t V_t \frac{\delta\rho}{r_t(u)} - \left[\frac{(\rho-1)(1-\delta\rho)}{n_t - r_t(u)\ (\rho-1)(1+bt)}\right]$$

$$(3.11) \qquad K_t = P_t(u)\ V_t\ \frac{\alpha(1-\delta\rho)}{n_t - r_t(u)\ (\rho-1)(1+bt)}$$

As $\alpha=1$, substituting equations (3.10) and (3.11) into (3.1), we have

$$(3.12) \qquad P_t(u) = A^{-1}e^{-\lambda t} \cdot [n_t - r_t(u) \cdot (\rho-1)(1+bt)]^{1-\delta\rho}$$

$$[1-\delta\rho]^{\delta\rho-1}\ [r_t(u)]^{\delta\rho} \cdot [\delta\rho]^{-\delta\rho}$$

From equation (3.12) the price elasticity of housing with respect to n and r(u) can be rewritten as

$$(3.13) \qquad \frac{\partial P_t(u)}{\partial n_t} \cdot \frac{n_t}{P_t(u)} = \left(n_t\right)[n_t - r_t(u)\ (\rho-1)(1+bt)]^{-1}\ (1-\delta\rho)$$

$$(3.14) \qquad \frac{\partial P_t(u)}{\partial r_t(u)} \cdot \frac{r_t(u)}{P_t(u)} =$$

$$\frac{[(\delta\rho)n_t r_t(u)^{\frac{\delta\rho}{1-\delta\rho}} - r_t(u)^{\frac{\delta\rho}{1-\delta\rho}+1} \cdot (1-\rho)(1+bt)]}{[n_t r_t(u)^{\frac{\delta\rho}{1-\delta\rho}} - r_t(u)^{\frac{\delta\rho}{1-\delta\rho}}\ (1-\rho)]}$$

Equations (3.13) and (3.14) allow the elasticities of housing services with respect to n_t and $r_t(u)$ to vary with changes in n_t and $r_t(u)$. These elasticities will reduce to $(1-\alpha)$ and α when the elasticity of substitution between land and capital

is unity. This is the case obtained with the Cobb-Douglas function.

Muth (14), using constrained utility maximization, has derived a differential equation to describe the relationship between the change in the cost of housing and the change in commuting cost as

$$P_t'(u)X_t d(u) + T = 0 \tag{3.15}$$

where $P_t'(u)$ is the slope of $P_t(u)$ and T is the cost of commuting. Substituting equation (3.4) into equation (3.12) we have

$$P_t'(u)Bw_t^{\Theta_1} p_t(u)^{\Theta_2} + T = 0 \tag{3.16}$$

Using the initial condition of equation (3.16) obtained from (3.12), the solution of the differential equation (3.16) can be written as

$$P_t(u) = \left[\bar{p}_t^{(\Theta_2+1)} + \frac{T(\bar{u}-u)\ (\Theta_2+1)}{Bw_t^{\Theta_1}}\right]^{\frac{1}{\Theta_2+1}} \tag{3.17}$$

where

$\bar{u}$ = the distance from the city center to the edge of the urban area, and

$\bar{P}_t = p_t(\bar{u})$

Substituting equation (3.12) into equation (3.17) and with some rearrangement we have

$$(3.18)\qquad \left[\bar{P}_t^{(\Theta_2+1)} + \frac{T(\bar{u}-u)\ (\Theta_2+1)}{Bw^{\Theta_1}}\right]^{\frac{1}{1+\Theta_2}}$$

$$= A^{-1}e^{-\lambda t}.\,[n_t - r_t(u)\ (\rho-1)(1+bt)]^{1-\delta\rho}$$

$$[1-\delta\rho]^{\delta\rho-1}\ [r_t(u)]^{\delta\rho}\ [\delta\rho]^{-\delta\rho}$$

where

$$(3.19)\quad \bar{P}(u) = A^{-1}e^{-\lambda t}\ [n_t - \bar{r}_t(u)\ (\rho-1)(1+bt)]^{1-\delta\rho}\ [1-\delta\rho]^{\delta\rho-1}$$

$$[\bar{r}_t(u)]^{\delta\rho}\ [\delta\rho]^{-\delta\rho}$$

Equation (3.18) can be written as

$$(3.20)\qquad r_t(u) = [A^{-1}e^{-\lambda t}(\delta\rho)^{-\delta\rho}(1-\delta\rho)^{\delta\rho-1}]^{1/\delta\rho}$$

$$n^{\left(\frac{1-\delta\rho}{\delta\rho}\right)}\left[\bar{P}_t^{(\Theta_2+1)} + \frac{t(\bar{u}-u)(\Theta_2+1)}{Bw^{\Theta_1}}\right]^{\left(\frac{1}{\delta\rho(\Theta_2+1)}\right)}$$

$$\left[1 - \frac{r_t(u)}{n_t}\ (\rho-1)(1+bt)\right]^{1-\delta\rho}$$

This implies that the explicit relationship between $r_t(u)$ and u cannot be derived unless ρ is equal to unity. If $\rho=1$, then equation (3.20) reduces to

$$(3.21)\qquad r_t(u) = [A^{-1}e^{-\lambda t}\delta^{-\delta}(1-\delta)^{\delta-1}]^{1/\delta}\, n^{\left(\frac{1-\delta}{\delta}\right)} [\bar{P}_t^{(\theta_2+1)} + \frac{t(\bar{u}-u)(\theta_2+1)}{Bw^{\theta_1}}]^{\frac{1}{\delta(\theta_2+1)}}$$

where

$$(3.22)\qquad \bar{P}_t = A^{-1}e^{-\lambda t}\,(1-\delta)^{\delta-1}\delta^{-\delta} r_t^{-\delta}\, n_t^{1-\delta}$$

and

$$\bar{r} = r(\bar{u})$$

This is identical to the general case presented by Mills (13) except for the time variable.

To derive the relationship between the population density and the rental of land [r(u)], Mills (13, p. 84) has defined

$$(3.23)\qquad N_t(u) = \frac{V_t(u)}{X_t d(u)}$$

where

$N_t(u)$ = the number of workers living at u;

$V_t(u)$ = output of housing services; and

$X_t d(u)$ = housing demand per worker at u.

From equations (3.10) and (3.11) it can be shown that

$$(3.24)\qquad K_t(u) = \frac{\left[\frac{1-\delta\rho}{n_t - r_t(u)\ (\rho-1)(1+bt)}\right]}{\frac{\delta\rho}{r_t(u)} - \frac{(\rho-1)(1-\delta\rho)}{n_t - r_t(u)\ (\rho-1)(1+bt)}}\ L_t(u)$$

Substituting equation (3.18) into (3.1)

$$(3.25)\quad V_t(u) = Ae^{\lambda t}L_t(u)\ \left[\frac{\delta\rho}{r_t(u)} - \frac{(\rho-1)(1-\delta\rho)}{n_t - r_t(u)\ (\rho-1)(1+bt)}\right]^{-1} \cdot$$

$$\left[\frac{1-\delta\rho}{n_t - r_t(u)\ (\rho-1)(1+bt)}\right] \cdot$$

$$\left[\frac{\delta\rho}{r_t(u)} - \frac{(\rho-1)(1-\delta\rho)(bt)}{n_t - r_t(u)\ (\rho-1)(1+bt)}\right]$$

From equation (3.4) and (3.12),

$$(3.26)\quad X_t d(u) = Bw^{\Theta_1}A^{-\Theta_2}e^{-\lambda\Theta_2 t}[n_t - r_t(u)\ (\rho-1)(1+bt)]^{\Theta_2(1-\delta\rho)}$$

$$[1-\delta\rho]^{\Theta_2(\delta\rho-1)}\quad [r_t(u)]^{\delta\rho\Theta_2}[\delta\rho]^{-\delta\rho\Theta_2}$$

Substituting (3.25) and (3.26) into (3.23)

$$(3.27)\quad D_t(u) = \frac{N_t(u)}{L_t(u)} = B^{-1}w^{-\Theta_1}A^{\Theta_2+1}e^{\lambda t(\Theta_2+1)}$$

$$[n_t - r_t(u)\ (\rho-1)(1+bt)]^{\Theta_2(\delta\rho-1)-1}$$

$$[1-\delta\rho]^{\Theta_2(1-\delta\rho)+1}\quad \left[\frac{\delta\rho}{r_t(u)} - \frac{(\rho-1)\ (1-\delta\rho)}{n_t - r_t(u)\ (\rho-1)\ (1+bt)}\right]^{-1}$$

$$\left[\frac{\delta\rho}{r_t(u)} - \frac{(\rho-1)\ (1-\delta\rho)\ (bt)}{n_t - r_t(u)\ (\rho-1)\ (1+bt)}\right]\quad r_t(u)^{-\delta\rho\Theta_2}[\delta\rho]^{\delta\rho\Theta_2}$$

This is the population density in terms of r, $r_t(u)$, b, t and other parameters. To compare this density function with those derived from Cobb-Douglas production function, a static population density model derived from equations (3.20) and (3.27) can be written as

$$(3.28)\quad D_t(u) = B^{-1} w^{-\theta_1} A[1-\delta\rho]^{\alpha\theta_2(1-\delta\rho)+1} [\delta\rho]^{\alpha\theta_2\delta\rho} n^{\alpha\theta_2(\delta\rho-1)-1}$$

$$\bar{p}^{(\theta_2+1)} + \frac{t(\bar{u}-u)(\theta_2+1)}{Bw^{\theta_1}}^{-\frac{\theta_2}{1+\theta_2}} \left[1 - \frac{(\rho-1)(1-\delta\rho)}{\delta\rho\left[\frac{n}{r(u)} - (\rho-1)\right]}\right]^{-1}$$

$$\left[1 - \frac{r(u)}{n}(\rho-1)\right]$$

If $\rho=1$, then (3.28) reduces to Mills' (13) generalized results. If ρ is not equal to zero, then n and r(u) are additional variables needed to explain the change of population density within a city. This essentially is due to the VES production function having explicitly taken the K/L ratio into account. The argument can explicitly be shown by rewriting the static version of VES as

$$(3.27)\quad Y(u) = A\, K(u)^{\alpha(1-\delta\rho)} L^{\alpha\delta\rho} \left[1 + (\rho-1)\frac{K(u)}{L(u)}\right]^{\alpha\delta\rho}$$

within a city. K/L ratio change-over distance, and therefore, K/L becomes an important factor in explaining the supply of housing service unless the value of $(1-\rho)$ is trivial. In sum, this

density gradient estimated by using negative exponential function is generally biased.

In the following section, a regression model (for particular time point) is derived for estimating the elasticity of substitution. Some empirical results are estimated to demonstrate the importance of the possible impact of changes in σ on the density gradient.

4. Empirical Estimates of the VES Function

The VES and CES production functions relate the output-capital ratio to factor prices. The major difference lies in the fact that the VES is linear and CES is log-linear. Assuming constant returns to scale, a stochastic model of the VES function may be written (Revankar [16]):

$$(4.1) \qquad \log V = b_0 + b_1 X^* + \varepsilon_1$$

where

$$X^* = (1-\delta\rho) \log K + \delta\rho \, \text{Log} \, [L + (\rho-1)K]$$

$$b_0 = \log A$$

$$b_1 = \alpha$$

and

$$(4.2) \qquad V/K = F_1 n + F_2 r + \varepsilon_2$$

where

$$F_1 = 1/q(1-\delta\rho)$$

$$F_2 = (1-\rho)/q(1-\delta\rho)$$

$$q = e^{\sigma/2}$$

n = price per unit of capital

r = price per unit of land.[5]

and

$$(4.3) \qquad L/K = G_0 + G_1(n/r) + \varepsilon_3$$

where

$$G_0 = (1-\rho)/(1-\delta\rho)$$

$$G_1 = \delta\rho/(1-\delta\rho)$$

Thus the estimation of the VES function results in the empirical determination of the parameters γ, α, δ, and ρ. These parameters are estimated using Zellner's [21] seemingly unrelated regression (SUR) technique. The simultaneous estimation of equation (4.2) and (4.3) results in efficient estimates for F_1, F_2, G_0 and G_1. Using these estimates, ρ and δ can be determined by

$$(4.4) \qquad \rho = 1 - (G_0/1+G_1)$$

and

$$(4.5) \qquad \delta\rho = G_1/(1+G_1)$$

and

$$(4.6) \qquad \delta = G_1/(1+G_1-G_0)$$

Using these estimates, equation (4.1) can be estimated.

The elasticity of substitution, σ, is obtained from

$$\sigma = 1 + [(\rho-1)/(1-\delta\rho)]\ [K/L]$$

$$(4.7) \qquad \sigma = 1 - G_0(K/L)$$

The VES and CES functions are generalizations of the Cobb-Douglas function. Thus, a test is necessary to determine the relevance of using a VES or CES function. This is done by testing whether $\rho=1$, i.e., $G_0 \neq 1$, provides empirical support

for the VES. This same test is provided for in the CES by testing if the slope coefficient is equal to unity in equations (4.14) and (4.15).

The seemingly unrelated regression estimates of equations (4.2) and (4.3) are (standard errors in parentheses):

$$(4.8) \qquad V/K = \underset{(.144)}{.717} + \underset{(.011)}{1.023}\ (r) + \underset{(.032)}{.678}\ (n)$$

$$(4.9) \qquad L/K = \underset{(.098)}{.681} + \underset{(.008)}{.086}\ (n/r)$$

Thus,

$$(4.10) \qquad \rho = 1 - (G_0/1+G_1) = .373$$

and at the mean K/L ratio,

$$(4.11) \qquad \sigma = 1 - [G_0(K/L)] = .514$$

The capital-land ratio in the sample ranged from 1.336 to .104. Thus the elasticity of substitution (equation 4.7) varies from .090 to .929. These empirical results tend to support the use of a VES function and lend significant insight into the range of the elasticity of substitution in an urban area.

The factor intensity parameter, δ, is equal to

$$(4.12) \qquad \delta = G_1/[1+G_1-G_0] = .212$$

The estimates of the production function in equation (4.1) without restricting the returns to scale parameter, α, to unity, is

$$(4.13) \qquad \log V = \underset{(.176)}{.270} + \underset{(.017)}{.930}\ (X^*)$$

with $\bar{R}^2 = .962$. Hence, Log $A = .270$ and the returns to scale parameter, $\alpha = .930$.

The marginal conditions for the CES production function are

$$\log (V/K) = F_1' + F_2' \log n \tag{4.14}$$

where

$$F_1' = (-\rho/(1-\rho))\log \delta - (1/(1/\rho))\log (1-\delta)$$

$$F_2' = 1/(1-\rho) = \sigma$$

and

$$\log (L/K) = G_0' + G_1' \log (n/r) \tag{4.15}$$

where

$$G_0' = (1/(1-\rho))\log (\delta/(1-\delta))$$

$$G_1' = 1/(1-\rho) = \sigma$$

Equations (4.14) and (4.15) were estimated using Zellner's efficient technique by restricting $F_2' = G_1' = \sigma$. These results are (standard errors in parentheses):

$$\log (V/K) = \underset{(.066)}{1.148} + \underset{(.026)}{.619} \log(n) \tag{4.16}$$

$$\log (L/K) = \underset{(.053)}{-.925} + \underset{(.026)}{.619} \log(n/r) \tag{4.17}$$

Thus, the elasticity of substitution is .62 in the CES case.[6]

5. Summary and Concluding Remarks

Based upon the VES production function, the land-use model is derived to show that both the Cobb-Douglas and CES production functions cannot take into account changing elasticity of substitution of capital for land. It is also shown that the ratio between rental rate of land and capital may be an additional exploratory variable for the density gradient function. The empirical results demonstrate that the VES production function rather than the

Cobb-Douglas or the CES should be used to investigate urban land use. The results indicate that the elasticity of substitution does change with distance in an urban area and that the use of a Cobb-Douglas or CES production function to derive a density gradient will result in biased estimates. A VES production function produces a density gradient function which decreases at a numerically faster rate with distance. In sum, this study not only has found a most useful application for the VES production function but also has shown that city planning (or urban structure determination) cannot entirely rely upon the distance variable.

Footnotes

[1]Other forms of a VES function have been developed [see Bruno (2); Liu and Hildebrand (9); Sato (18)]. Revankar's VES production function was selected since it is a generalized case of the Cobb-Douglas and it also allows us to discriminate the VES from the CES on statistical ground.

[2]The estimated relationship using the Ann Arbor data was $\ln(K/L) = .247 - .122 \ln(u)$ (standard errors in parentheses)
(.032) (.007)
with $\bar{R}^2 = .75$. The price of capital relative to the price of land increased with distance as indicated by:
$\ln(n/r) = .969 + .953 \ln(u)$
(.053) (.041)
with $\bar{R}^2 = .82$.

[3]Data in this study are for multi-family dwellings only. Therefore it is assumed for the empirical estimates that the production function is the same for different outputs and firms.

[4]The total value of capital relative total land varied with respect to distance by:
$\ln(nK/rL) = 1.379 + .392 \ln(u)$
(.053) (.042)
with $\bar{R}^2 = .42$. This would be expected with the elasticity (σ) being less than one.

[5]Appreciation is extended to Roger Koenker for making his data available. This data, used by Koenker [8], is based on the tax assessment records for all private multi-family housing constructed in Ann Arbor, Michigan, from 1964 through 1966. The sample consisted of 122 observations. It was assumed that there is no systematic bias in the assessment estimates of actual market values of the properties and the sites. The price of land and capital was determined as the price per square foot. The size of the building site and the number of units were contained in the data set.

[6]To discriminate the CES production function from VES production function, Lovell [10] has employed Zarembka' [20] functional form approach to test the estimated function form parameter (λ) for the relation between factor proportions and factor price ratio. Following Lovell's approach we have found that $\lambda = .6$ for our case. By employing the χ^2 test, it is found that λ is significantly different from both one and zero at the 5% significant level. However, under 1% significant level, it is found that λ is significantly different for zero and not significantly different for one. This result tends to support the VES function.

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